General motivation:

G finite group, k field, $\mathrm{char}\, k \mid |G|$ $\rightsquigarrow$ $\mathrm{mod}\, kG$
wanna understand!

Ex: C_p $k[x]/(x^p)$ indecomposables are $k[x]/(x^i)$ $(1 \leq i < p)$
$\rightsquigarrow$ all indec are parametrized.

Ex: $V = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ $\rightsquigarrow$ kV is tame, so $\exists$ param. too...

In general, kG is "wild", i.e.: cannot classify indec. modules (in a precise sense!)

? What can we do? (don't wanna give up!)

For example, can try to parametrize (nice) subcategories of $\mathrm{Mod}\, kG$ / $D^b(kG)$

"nice" means: thick subcategories, or thick $\otimes$-ideals:

$\mathcal{L} \subseteq D^b(kG)$ is thick if it is a full Δ'ed subcat.
- $\mathcal{L}[i] = \mathcal{L}$ $\forall i \in \mathbb{Z}$
- if $X, Y \in \mathcal{L}$, $X \xrightarrow{f} Y$ $\Rightarrow$ $\mathrm{Cone}(f) \in \mathcal{L}$
- If $X \oplus X' \in \mathcal{L}$ $\Rightarrow$ $X, X' \in \mathcal{L}$

Moreover, $\mathcal{L}$ a $\otimes$-ideal if:
- $\forall X \in D^f(kG)$, $Y \in \mathcal{L}$ $\Rightarrow$ $X \otimes Y \in \mathcal{L}$.

Let: $\mathrm{Thick}^{\otimes}(D^b(kG))$ denote the lattice of thick $\otimes$-ideals
(poset by inclusion, meet $\wedge$ is intersection $\cap$)

Q: Describe $D^b(kG)$?

Why should we do this?

$D^b(kG)$ is Δ'ed, so $X \xrightarrow{f} \Sigma^i X$ $\rightsquigarrow$ $X \xrightarrow{f} \Sigma^i X \to Y \to \Sigma X$
$\Rightarrow$ $Y \in \mathrm{thick}(X) :=$ smallest thick subcat. containing X.

. If $S \subseteq D^b(kG)$ a set of objects, can build inductively:

$$\langle S\rangle_1 := \mathrm{add}(\Sigma^i S \mid i \in \mathbb{Z}) \quad \text{(sums \& summands)}$$

$$\langle S\rangle_{i+1} := \mathrm{add}(Y \mid \exists\, \Delta \quad X_i \to Y \to X_1 \to \Sigma X_i), \quad X_i \in \langle S\rangle_i,\ X_1 \in \langle S\rangle_1$$

$$\langle S\rangle_1 \subseteq \cdots \subseteq \langle S\rangle_{i+1} \subseteq \cdots$$

Observation: $\mathrm{thick}(S) = \bigcup_{i\geq 1} \langle S\rangle_i$

So, this says if $X \in \mathrm{thick}(S)$, $\exists$ some sequence of Δ's and retracts describing how X is built from obj's in $\langle S\rangle_1$.

"$\exists$ finite resolution of X by objects of S".

<u>E.g.</u>: $\mathrm{Thick}(kG) = D^{\mathrm{perf}}(kG) \simeq K^b(\mathrm{proj}\, kG)$

<u>Obs</u>: If S is $\otimes$-closed i.e. $\forall\, X \in D^b(kG)$, $X \otimes S \subseteq S$

$\Rightarrow$ $\mathrm{thick}(S)$ is a $\otimes$-ideal. (Exercise!)

In particular: $S := \{X \otimes M \mid \forall\, X \in D^b(kG)\}$ (M fixed)

$\rightsquigarrow$ $\mathrm{Thick}(S) = \mathrm{thick}^{\otimes}(M)$, the thick $\otimes$-id. generated by M.

<u>Useful</u>: ($\exists$ many properties preserved by taking cones, summands etc so stable on the $\mathrm{thick}^{(\otimes)}(M)$ gen. by M.

History:

- Essentially begins with the nilpotence theorem of Devinatz-Hopkins-Smith (→ see Jan's talk)

Gives a description of the lattice Thick($SH_{(p)}$). finite spectra

- Using these ideas, the following thm.:

<u>Thm</u>. (Hopkins – Neeman)

R commutative noetherian ring, $\exists$ lattice isomorphism

$$\underset{\substack{\uparrow\simeq \\ K^b(\mathrm{proj}\,R)}}{\mathrm{Thick}\; D^{\mathrm{perf}}(R)} \cong \left\{ \begin{array}{l} \text{specialization closed subsets} \\ \text{of } \mathrm{Spec}(R) \end{array} \right\}$$

V spc-closed :⇔ if $\mathfrak{p} \subseteq \mathfrak{q}$, $\mathfrak{p} \in V \Rightarrow \mathfrak{q} \in V$

(i.e $V = \bigcup_\alpha V_\alpha$ with V_α closed)

Bijection given by support, i.e localize at a $\mathfrak{p}$ and see if acyclic ...

- The analog of this theorem for the stable module category was proved by Benson-Carlson-Rickard

Very different proof! uses <u>Mod</u> kG ← ∞-generated kG-modules very essentially

- Carlson-Iyengar: gives a proof of previous result using nilpotence.

They use $D^b(kG)$ rather than <u>mod</u> kG, but Rickard's thm gives a direct connection, saying it's ≈ the same problem!

Thm. (BCR) We have a picture describing $\otimes$-ideals:

$$\begin{tikzcd}
D^{perf}(kG) \arrow[r, hook] \arrow[d, leftrightarrow, "\substack{\text{thick given}\\ \text{by spec.}\\ \text{subsets}}"'] & D^b(kG) \arrow[r, two heads, "\text{Rickard's} \simeq"] \arrow[d, leftrightarrow] & \underline{\mathrm{mod}}\, kG \arrow[d, leftrightarrow] \\
\{pt\} \arrow[r, hook, "\substack{\text{unique}\\ \text{closed pt}}"] & \mathrm{Spec}^h H^*(G;k) & \mathrm{Proj}\, H^*(G;k) \arrow[l, hook']
\end{tikzcd}$$

In each case, the thick $\otimes$-ideals correspond to special. closed subsets

homog. spectrum of the ~~co~~ cohom.-ring

Outline of the proof:

simpler

$D^b(kG)$

① ↓

$D^b(kE)$ — E elem.-abelian p-group

② ↓

$D^f(K)$ — K is the Koszul complex of kE (a dga)

③ ↓

$D^f(\Lambda)$ — Λ graded exterior algebra (fin.-dim.)

④ ↓

$D^f(S)$ — S graded polynomial rings (honest commutative setting!)

Λ, S: viewed as dga's

$D^f(?)$ = "bounded derived cat. of dg-modules over ?"

here can use the original $\otimes$-nilp. ideas to prove the Thm.

(1) Reduce to elem. abelian groups using Carlson's generation theorem.

$E < G$ elab subgroups, $\mathrm{Thick}^{\otimes}(E) \overset{\text{control}}{\rightsquigarrow} \mathrm{Thick}^{\otimes}(G)$

Bregje — Sebastian

(2) If $E = (\mathbb{Z}/p\mathbb{Z})^{\oplus r}$ then $kE \cong k[z_1, \dots, z_r]/(z_1^p, \dots, z_r^p)$

$\rightsquigarrow K := (kE \xrightarrow{z_1} kE) \otimes \dots \otimes (kE \xrightarrow{z_r} kE)$, Koszul complex

Jesse: has a dga structure

$\rightsquigarrow \exists$ iso $\quad \mathrm{Thick}^{\otimes} D^b(kE) \cong \mathrm{Thick}\, D^f(K)$,

(3) K has cohomology a graded exterior alg Λ on r gen's.

and $\quad \Lambda = H^*(K) \xrightarrow{\sim} K$ is a quasi-iso of dga's

$\Longrightarrow \quad D^f(K) \simeq D^f(\Lambda)$ as a dga with $d = 0$.

(4) We can use the BGG correspondence (Andrea) to get $\quad D^f(\Lambda) \cong D^f(S)$.

After all this, one has $\quad \mathrm{Thick}^{(\otimes)} D^b(kE) \cong \mathrm{Thick}^{(\otimes)} D^f(S)$

Finally use a nilpotence theorem + ideas from support varieties to prove that

$$\mathrm{Thick}\, D^f(S) \cong \left\{ \begin{array}{c} \text{spc-closed subsets} \\ \text{of } \mathrm{Spec}^h S \end{array} \right\}$$

Sira

Observation:

$\mathrm{Spec}^h H^*(E;k) \cong \mathrm{Spec}^h(S)$ a sym. polyn. ring, as a central subring of $H^{\otimes}(G,k)$

el. abelian

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